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Gérard-Levelt Membranes

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Introduction

Given a meromorphic linear differential system on the Riemann sphere,

$$\frac{dX}{dz} = A(z)X \text{ with } A(z) \in \mathrm{M}_n(\mathbb{C}(z)), \tag{1}$$

the nature of a singularity of A can be tackled by the study of lattices in a connection attached to the differential system. We use the geometric framework of the Bruhat-Tits building of $\mathrm{SL}(K)$, where $K = \mathbb{C}((z))$, and show that this can be performed by computations on the tropical linear space L_p attached to the valuated matroid p corresponding to a given *membrane* in the Bruhat-Tits building.

Meromorphic connections

A *meromorphic connection* is a map $\nabla : V \simeq K^n \longrightarrow \Omega(V) = V \otimes_K \Omega^1_{\mathbb{C}}(K)$ which is $\mathbb{C}$-linear and satisfies the Leibniz rule

$$\nabla(fv) = v \otimes df + f\nabla v \text{ for } f \in K \text{ and } v \in V.$$

The *matrix* $\mathrm{Mat}(\nabla,(e))$ is given by $\nabla e_j = -\sum_{i=1}^n e_i \otimes \Omega_{ij}$ for a basis (e). A basis change $P \in \mathrm{GL}_n(K)$ *gauge-transforms* the matrix of ∇ by

$$\Omega_{[P]} = P^{-1}\Omega P - P^{-1}dP. \tag{2}$$

Contracting with $z^k\frac{d}{dz}$ yields a differential operator ∇_k, and system (1) is the expression of $\nabla_0(v) = 0$ in the basis (e).
A *lattice* Λ in V is a free sub-$\mathcal{O}$-module of rank n, that is a module of the form

$$\Lambda = \bigoplus_{i=1}^n \mathcal{O}e_i \text{ for some basis } (e) \text{ of } V.$$

The *Poincaré rank of* ∇ *on the lattice* Λ is defined as the integer

$$\mathfrak{p}_\Lambda(\nabla) = -v_\Lambda(\Lambda + \nabla_1(\Lambda)) = \min\{k \in \mathbb{N} \mid \nabla_{k+1}(\Lambda) \subset \Lambda\} = \max_{i,j}(-v(\Omega_{ij}), 0). \tag{3}$$

The *true Poincaré rank* $m(\nabla) = \min_\Lambda \mathfrak{p}_\Lambda(\nabla)$ characterises the kind of solutions around the singularity of (1)

- $m(\nabla) = 0$, *regular* : $Y = U(z)z^L$ with U meromorphic and $L \in \mathrm{M}_n(\mathbb{C})$,
- $m(\nabla) > 0$, *irregular* : Y asymptotic to $\hat{F}(z)z^L e^Q$, with $\hat{F}$ meromorphic formal, $Q = \mathrm{diag}(q_1(z^{1/k}), \ldots, q_n(z^{1/k}))$ with $q_i \in \frac{1}{z}\mathbb{C}[\frac{1}{z}]$.

Another important invariant is the *Katz rank* $\kappa(\nabla) = \frac{1}{k}\max_i(-\deg q_i)$.

Example: $A = \begin{pmatrix} z^{-1} & z^{-N} \\ 0 & z^{-1} \end{pmatrix}$ has Poincaré rank $\mathfrak{p} = N-1$ but is regular.

Gérard-Levelt's saturated lattices

For $k \geqslant 1$, Gérard and Levelt define the lattices

$$F_k^\ell(\Lambda) = \Lambda + \nabla_k\Lambda + \cdots + \nabla_k^\ell\Lambda.$$

Theorem 1 (Gérard, Levelt) *The true Poincaré rank* $m(\nabla)$ *of* ∇ *is*

$$m(\nabla) = \min\{k \in \mathbb{N} \mid F_k^{n-1}(\Lambda) \text{ has Poincaré rank } \leqslant k\} \text{ for any lattice } \Lambda \subset V.$$

Finding the true Poincaré rank is finding the largest lattice whose Poincaré rank matches its index in the following sequence

$$\Lambda \subset F_{p-1}^{n-1}(\Lambda) \subset \cdots \subset F_0^{n-1}(\Lambda) \tag{4}$$

The affine building of $\mathrm{SL}(V)$

The affine building B_n attached to $\mathrm{SL}(V)$ is the *flag simplicial complex* of the graph whose

- **vertices** are the homothety classes of lattices in V
- **edges** connect vertices L and L' for which $\exists\Lambda \in L, \exists M \in L'$ such that $z\Lambda \subset M \subset \Lambda$.

Let $M = \{d_1, \ldots, d_m\}$ be lines such that $d_1 + \cdots + d_m = V$. The subcomplex

$$[M] = \{\Lambda = \ell_1 + \cdots + \ell_m \mid \ell_i \text{ is a lattice in } d_i\}$$

is called by Keel and Tevelev the *membrane* spanned by M.
For a choice $\mathcal{A} = (v_1, \ldots, v_m)$ of non-zero vectors in the lines d_i, any lattice Λ in the membrane $[M]$ for $M = \{d_1, \ldots, d_m\}$ can be written $\Lambda_u = \sum_{i=1}^m \mathcal{O}z^{-u_i}v_i$ for a lattice point $u \in \mathbb{Z}^m$.

Tropical convexity and lattices

Membranes spanned by m lines in the Bruhat-Tits building have a faithful representation as tropical linear spaces in m-dimensional space.
Let $(\mathbb{T} = \mathbb{R} \cup \{\infty\}, \oplus = \min, \odot)$ be the tropical semialgebra and consider the *projective tropical spaces*

$$\mathbb{TA}^{m-1} = \mathbb{R}^m/\mathbb{R}(1,\ldots,1) \text{ and } \mathbb{TP}^{m-1} = \mathbb{T}^m \backslash \{(\infty,\ldots,\infty)\}/\mathbb{R}(1,\ldots,1).$$

A membrane M and a basis (e) of V determine a *valuated matroid*

$$\begin{aligned} p : [m]^n &\longrightarrow \mathbb{R} \cup \{\infty\} \\ \omega &\longmapsto v(\det_{(e)} M_\omega) \end{aligned}$$

where $M_\omega = (v_{\omega_1}, \ldots, v_{\omega_n})$ is the subfamily of vectors of M indexed by ω.
A *cocircuit* of the matroid p is a vector defined for some $\sigma \in \binom{[m]}{n-1}$ by

$$p(\sigma*) = (p(\sigma \cup \{1\}), \ldots, p(\sigma \cup \{m\})) \in \mathbb{T}^m$$

Theorem 2 (Yu,Yuster) *The* tropical linear space $L_p \subset \mathbb{TP}^{m-1}$ associated with this valuated matroid *is the tropical convex hull of the cocircuits of* p.

Theorem 3 (Keel,Tevelev) *The nearest point projection map* π_{L_p} : $\mathbb{TP}^{m-1} \longrightarrow L_p$ *be induces a bijection* Ψ_M *between* $[M]$ *and* L_p

$$\Psi_M(\Lambda_u) = \pi_{L_p}(u_1, \ldots, u_m).$$

In particular, $M_u = (z^{-u_1}v_1, \ldots, z^{-u_m}v_m)$ and $M_{u'} = (z^{-u'_1}v_1, \ldots, z^{-u'_m}v_m)$ span the same lattice Λ (up to homothety) if and only if $\pi_{L_p}(u) \equiv \pi_{L_p}(u')$ modulo tropical scalar multiplication.
There are explicit formulæ for this projection (Blue and Red rules – Ardila, Joswig, Sturmfels and Yu): if the projected points are computed by these rules, then we even have

$$\Lambda_u = \Lambda_{u'} \iff \pi_{L_p}(u) = \pi_{L_p}(u'). \tag{5}$$

The Gérard-Levelt membranes

Proposition 1 (C) *Fix a basis* (e) *of* Λ, *and* $\ell \geqslant 0$. *Let* $[M_\ell]$ *be the membrane spanned by the vectors* $(\nabla_1^j e_i)_{1\leqslant i\leqslant n, 0\leqslant j\leqslant \ell}$. *Then* $F_k^\ell(\Lambda) \in [M_\ell]$ *for all* $k \geqslant 0$.

The lattices $F_k^\ell(\Lambda)$ for $0 \leqslant \ell \leqslant n$ can all be seen as elements of the same membrane $[M_n]$. Indeed, $F_k^\ell(\Lambda)$ is represented by the lattice point

$$u_k^\ell = (\underbrace{0,\ldots,0}_{n \text{ times}}, \underbrace{k,\ldots,k}_{n \text{ times}}, \ldots, \underbrace{k\ell,\ldots,k\ell}_{n \text{ times}}, v_\Lambda(\nabla_1^{\ell+1}e_1), \ldots, v_\Lambda(\nabla_1^n e_n)).$$

Definition 1 $\mathcal{M}_{\mathrm{GL}}^\Lambda = [M_n]$ *is called the Gérard-Levelt membrane attached to* Λ.

If $\mathrm{Mat}(\nabla_1,(e)) = A$, in this basis $\mathcal{M}_\Lambda$ is described by the $n \times n(n+1)$ matrix

$$\mathrm{M} = \begin{pmatrix} I_n & A & \cdots & A_n \end{pmatrix} \text{ where } A_{k+1} = (z\tfrac{d}{dz} + A)A_k \text{ and } A_0 = I_n.$$

The tropical projection $\pi_{\mathrm{GL}}^\Lambda$ onto the tropical linear space L_{GL}^Λ attached to the Gérard-Levelt membrane $\mathcal{M}_{\mathrm{GL}}^\Lambda$ maps a point u to a *unique* representative. Checking if $k \geqslant m(\nabla)$ requires to know if the lattice points u_k^{n-1} and u_k^n represent the same lattice, that is, by (5)

$$\pi_{\mathrm{GL}}^\Lambda(u_k^n) = \pi_{\mathrm{GL}}^\Lambda(u_k^{n-1}).$$

Corollary 1 *For any* Λ, *we have* $m(\nabla) = \min\{k \in \mathbb{N} \mid \pi_{\mathrm{GL}}^\Lambda(u_k^n) = \pi_{\mathrm{GL}}^\Lambda(u_k^{n-1})\}$.

An intriguing example

$$A = \begin{pmatrix} -5z^{-2} & 5z^{-1} & -2z^{-1} & -9z^{-2} \\ 5z^{-3} & 3z^{-2} & 2z^{-2} & -4z^{-2} \\ 4z^{-1} & -5z^{-1} & -5z^{-2} & 2 \\ \frac{2-2z}{z^3} & -5z^{-1} & 3z^{-2} & -6z^{-2} \end{pmatrix}.$$

$$u_k^n = (0,\ldots,-3k,-4k-4k-4k-4k)$$
$$u_k^{n-1} = (0,\ldots,-3k,-6,-5,-5,-6).$$

$$\text{One gets } \pi_{\mathrm{GL}}(u_k^n) = \pi_{\mathrm{GL}}(u_k^{n-1}) \iff k \geqslant \frac{3}{2}, \text{ therefore } m(\nabla) = 2.$$

The Katz rank is usually computed after ramifying the variable $\zeta = z^{1/p}$ to a suitable order. Here $\kappa(\nabla) = \frac{3}{2}$, so the formula in corollary 1 generalizes to $\mathbb{R}^+$ in this case.

Perspectives

- Which other invariants of connections admit tropical definitions and/or computations?
- Is there a computation of the tropical projection map which is more efficient than the methods based on gauge transformations?

References

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